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A new generalization of some integral inequalities for (α, m)-convex functions

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Abstract

In this paper, we derive new estimates for the remainder term of the midpoint, trapezoid, and Simpson formulae for functions whose derivatives in absolute value at certain power are (α, m)-convex.

Keywords: (α, m)-Convex function, Hermite-Hadamard's inequality, Simpson type inequality, Trapezoid inequality, Midpoint inequality

Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality is well known in the literature as Hermite-Hadamard integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

The class of (α, m)-convex functions was first introduced in [1], and it is defined as follows:

The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (α, m)-convex where $(\alpha, m) \in [0, 1]^2$, if we have

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. It can be easily deduced that for $(\alpha, m) \in \{(0,0), (\alpha,0), (1,0), (1,m), (1,1), (\alpha,1)\}$ one obtains the following classes of functions: increasing, α star-shaped, star-shaped, m-convex, convex, and α-convex functions, respectively.

As denoted by $K_m^{\alpha}(b)$, the set of all (α, m)-convex functions is on $[0, b]$ for which $f(0) \leq 0$. For recent results and generalizations concerning (α, m)-convex functions, see [1-8].

In [7], Set et al. proved the following Hadamard type inequality for (α, m)-convex functions.

Theorem 1. *Let $f : [0, \infty) \to \mathbb{R}$ be an (α, m)-convex function with $(\alpha, m) \in (0, 1]^2$. If $0 \leq a < b < \infty$ and $f \in L[a, b]$, then one has the inequality*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \min\left\{\frac{f(a) + \alpha m f\left(\frac{b}{m}\right)}{\alpha+1}, \frac{f(b) + \alpha m f\left(\frac{a}{m}\right)}{\alpha+1}\right\}. \tag{2}$$

The following inequality is well known in the literature as Simpson's inequality.

Let $f : [a, b] \to \mathbb{R}$ be a four-time continuously differentiable mapping on (a, b) and $\left\|f^{(4)}\right\|_{\infty} = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then the following inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.$$

In recent years, many authors have studied error estimations for Simpson's inequality; for refinements, counterparts, generalizations, and new Simpson type inequalities, see [9-12].

In this paper, in order to provide a unified approach to establish midpoint, trapezoid, and Simpson type inequality for functions whose derivatives in absolute value at certain power are (α, m)-convex, we derive a general integral identity for convex functions.

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Main results

In order to generalize the classical Trapezoid, midpoint and Simpson type inequalities and prove them, we need the following Lemma:

Lemma 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$, $\lambda, \mu \in [0,1]$ and $m \in (0,1]$. Then the following equality holds:*

$$\lambda\left(\mu f(a)+(1-\mu)f(mb)\right)+(1-\lambda)f(\mu a+m(1-\mu)b) - \frac{1}{mb-a}\int_a^{mb} f(x)dx = (mb-a) \times \left[\int_0^{\mu}[-t+\lambda(1-\mu)]f'(ta+m(1-t)b)\,dt + \int_\mu^1 [-t+(1-\mu\lambda)]f'(ta+m(1-t)b)\,dt\right]. \tag{3}$$

A simple proof of the equality can be done by performing an integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader.

Theorem 2. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a < b$ and $\lambda,\mu \in [0,1]$. If $|f'|^q$ is (α,m)-convex on $[a,b]$, for $(\alpha,m) \in (0,1]^2$, $mb > a$, $q \geq 1$, then the following inequality holds:*

$$\left|\lambda\left(\mu f(a)+(1-\mu)f(mb)\right)+(1-\lambda)f(\mu a+m(1-\mu)b) - \frac{1}{mb-a}\int_a^{mb} f(x)dx\right| \tag{4}$$

$$\leq \begin{cases} (mb-a)\left\{\varepsilon_2^{1-\frac{1}{q}}\left(\delta_3|f'(a)|^q+m\delta_4|f'(b)|^q\right)^{\frac{1}{q}} + \varepsilon_3^{1-\frac{1}{q}}\left(\beta_1|f'(a)|^q+m\beta_2|f'(b)|^q\right)^{\frac{1}{q}}\right\} & ,\ \lambda(1-\mu)\leq\mu\leq 1-\lambda\mu \\ (mb-a)\left\{\varepsilon_1^{1-\frac{1}{q}}\left(\delta_1|f'(a)|^q+m\delta_2|f'(b)|^q\right)^{\frac{1}{q}} + \varepsilon_3^{1-\frac{1}{q}}\left(\beta_1|f'(a)|^q+m\beta_2|f'(b)|^q\right)^{\frac{1}{q}}\right\} & ,\ \mu\leq\lambda(1-\mu)\leq 1-\lambda\mu \\ (mb-a)\left\{\varepsilon_2^{1-\frac{1}{q}}\left(\delta_3|f'(a)|^q+m\delta_4|f'(b)|^q\right)^{\frac{1}{q}} + \varepsilon_4^{1-\frac{1}{q}}\left(\beta_3|f'(a)|^q+m\beta_4|f'(b)|^q\right)^{\frac{1}{q}}\right\} & ,\ \lambda(1-\mu)\leq 1-\lambda\mu\leq\mu \end{cases},$$

where

$$\varepsilon_1 = -\frac{\mu^2}{2}+\lambda(1-\mu)\mu,\ \varepsilon_2=[\lambda(1-\mu)]^2-\varepsilon_1,$$

$$\varepsilon_3=(1-\lambda\mu)^2-(1-\lambda\mu)(1+\mu)+\frac{1+\mu^2}{2},$$

$$\varepsilon_4=\frac{1-\mu^2}{2}-(1-\lambda\mu)(1-\mu),$$

$$\delta_1=\frac{\lambda(1-\mu)\mu^{\alpha+1}}{\alpha+1}-\frac{\mu^{\alpha+2}}{\alpha+2},$$

$$\delta_2=\lambda(1-\mu)\mu-\frac{\mu^2}{2}-\delta_1,$$

$$\delta_3=\frac{2[\lambda(1-\mu)]^{\alpha+2}}{(\alpha+1)(\alpha+2)}-\frac{\lambda(1-\mu)\mu^{\alpha+1}}{\alpha+1}+\frac{\mu^{\alpha+2}}{\alpha+2},$$

$$\delta_4=[\lambda(1-\mu)]^2-\lambda\mu(1-\mu)+\frac{\mu^2}{2}-\delta_3,$$

$$\beta_1=\frac{2(1-\lambda\mu)^{\alpha+2}}{(\alpha+1)(\alpha+2)}-\frac{(1-\lambda\mu)\left(1+\mu^{\alpha+1}\right)}{\alpha+1}+\frac{1+\mu^{\alpha+2}}{\alpha+2},$$

$$\beta_2=(1-\lambda\mu)^2-(1-\lambda\mu)(1+\mu)+\frac{1+\mu^2}{2}-\beta_1,$$

$$\beta_3=\frac{1-\mu^{\alpha+2}}{\alpha+2}-(1-\lambda\mu)\frac{1-\mu^{\alpha+1}}{\alpha+1},$$

$$\beta_4=(1-\lambda\mu)(\mu-1)+\frac{1-\mu^2}{2}-\beta_3.$$

Proof. From Lemma 1 and using the properties of modulus and the well known power mean inequality, we have

$$\left|\lambda\left(\mu f(a)+(1-\mu)f(mb)\right)+(1-\lambda)f(\mu a+m(1-\mu)b)\right.$$
$$\left.-\frac{1}{mb-a}\int_a^{mb}f(x)dx\right|$$
$$\leq (mb-a)\left[\int_0^{\mu}|-t+\lambda(1-\mu)|\left|f'(ta+m(1-t)b)\right|dt\right.$$
$$\left.+\int_{\mu}^{1}|-t+(1-\mu\lambda)|\left|f'(ta+m(1-t)b)\right|dt\right]$$
$$\leq (mb-a)\left\{\left(\int_0^{\mu}|-t+\lambda(1-\mu)|\,dt\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(\int_0^{\mu}|-t+\lambda(1-\mu)|\left|f'(ta+m(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\mu}^{1}|-t+(1-\lambda\mu)|\,dt\right)^{1-\frac{1}{q}}$$
$$\left.\times\left(\int_{\mu}^{1}|-t+(1-\lambda\mu)|\left|f'(ta+m(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\} \tag{5}$$

Since $|f'|^q$ is (α,m)-convex on $[a,b]$, we know that for $t\in[0,1]$

$$\left|f'(ta+m(1-t)b)\right|^q\leq t^{\alpha}\left|f'(a)\right|^q+m(1-t^{\alpha})\left|f'(b)\right|^q,$$

hence, by simple computation

$$\int_0^{\mu}|-t+\lambda(1-\mu)|\,dt=\begin{cases}\varepsilon_1, & \mu\leq\lambda(1-\mu)\\ \varepsilon_2, & \mu\geq\lambda(1-\mu)\end{cases}, \tag{6}$$

$$\int_{\mu}^{1}|-t+(1-\lambda\mu)|\,dt=\begin{cases}\varepsilon_3, & \mu\leq 1-\lambda\mu\\ \varepsilon_4, & \mu\geq 1-\lambda\mu\end{cases}, \tag{7}$$

$$\int_0^{\mu}|-t+\lambda(1-\mu)|\left|f'(ta+m(1-t)b)\right|^q dt$$
$$\leq\int_0^{\mu}|-t+\lambda(1-\mu)|\left[t^{\alpha}\left|f'(a)\right|^q+m(1-t^{\alpha})\left|f'(b)\right|^q\right]dt$$
$$=\begin{cases}\delta_1\left|f'(a)\right|^q+m\delta_2\left|f'(b)\right|^q, & \mu\leq\lambda(1-\mu)\\ \delta_3\left|f'(a)\right|^q+m\delta_4\left|f'(b)\right|^q, & \mu\geq\lambda(1-\mu)\end{cases}, \tag{8}$$

$$\int_{\mu}^{1}|-t+(1-\lambda\mu)|\left|f'(ta+m(1-t)b)\right|^q dt$$
$$\leq\int_{\mu}^{1}|-t+(1-\lambda\mu)|\left[t^{\alpha}\left|f'(a)\right|^q+m(1-t^{\alpha})\left|f'(b)\right|^q\right]dt$$
$$=\begin{cases}\beta_1\left|f'(a)\right|^q+m\beta_2\left|f'(b)\right|^q, & \mu\leq 1-\lambda\mu\\ \beta_3\left|f'(a)\right|^q+m\beta_4\left|f'(b)\right|^q, & \mu\geq 1-\lambda\mu\end{cases}. \tag{9}$$

Thus, using (6)-(9) in (5), we obtain the inequality (4). This completes the proof. □

Corollary 2.1. *Under the assumptions of Theorem 2 with $q=1$, we have*

$$\left|\lambda\left(\mu f(a)+(1-\mu)f(mb)\right)+(1-\lambda)f(\mu a+m(1-\mu)b)\right.$$
$$\left.-\frac{1}{mb-a}\int_a^{mb}f(x)dx\right|$$
$$\leq\begin{cases}(mb-a)\left\{(\delta_3+\beta_1)\left|f'(a)\right|+m(\delta_4+\beta_2)\left|f'(b)\right|\right\}, \\ \quad\lambda(1-\mu)\leq\mu\leq 1-\lambda\mu\\ (mb-a)\left\{(\delta_1+\beta_1)\left|f'(a)\right|+m(\delta_2+\beta_2)\left|f'(b)\right|\right\}, \\ \quad\mu\leq\lambda(1-\mu)\leq 1-\lambda\mu\\ (mb-a)\left\{(\delta_3+\beta_3)\left|f'(a)\right|+m(\delta_4+\beta_4)\left|f'(b)\right|\right\}, \\ \quad\lambda(1-\mu)\leq 1-\lambda\mu\leq\mu\end{cases}.$$

Remark 1. In Corollary 2.1,

(i) If we choose $\mu=\frac{1}{2}$, $\lambda=\frac{1}{3}$ and $\alpha=1$, we have the following inequality

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+mb}{2}\right)+f(mb)\right]-\frac{1}{mb-a}\int_a^{mb}f(x)dx\right|$$
$$\leq\frac{5}{72}(mb-a)\left\{\left|f'(a)\right|+m\left|f'(b)\right|\right\},$$

which is the same of the Simpson type inequality in Corollary 2.3 (ii) in [9].

(ii) If we choose $\mu=\frac{1}{2}$, $\lambda=1$ and $\alpha=1$, we have the following inequality

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)dx\right|$$
$$\leq\frac{(mb-a)}{8}\left\{\left|f'(a)\right|+m\left|f'(b)\right|\right\}$$

which is the same of the trapezoid type inequality in Corollary 2.3 (i) in [9].

(iii) If we choose $\mu=\frac{1}{2}$, $\lambda=0$ and $\alpha=1$, we have the following midpoint type inequality

$$\left|f\left(\frac{a+mb}{2}\right)-\frac{1}{mb-a}\int_a^{mb}f(x)dx\right|$$
$$\leq\frac{(mb-a)}{8}\left\{\left|f'(a)\right|+m\left|f'(b)\right|\right\}.$$

Corollary 2.2. *Under the assumptions of Theorem 2 with $\mu = \frac{1}{2}$ and $\lambda = \frac{1}{3}$, we have the following Simpson type inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+mb}{2}\right)+f(mb)\right]-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right|$$

$$\leq (mb-a)\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\left\{\left(\delta_3^*\left|f'(a)\right|^q+m\left(\frac{5}{72}-\delta_3^*\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\beta_1^*\left|f'(a)\right|^q+m\left(\frac{5}{72}-\beta_1^*\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\},$$

where

$$\delta_3^* = \frac{2+3^{\alpha+1}(2\alpha+1)}{6^{\alpha+2}(\alpha+1)(\alpha+2)}$$

and

$$\beta_1^* = \frac{2\times 5^{\alpha+2}+6^{\alpha+1}(\alpha-4)-3^{\alpha+1}(2\alpha+7)}{6^{\alpha+2}(\alpha+1)(\alpha+2)}.$$

Remark 2. In Corollary 2.2, if we take $\alpha = m = 1$, we obtain the following Simpson type inequality

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^{b} f(x)dx\right|$$

$$\leq (b-a)\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\left\{\left(\frac{29}{1296}\left|f'(b)\right|^q+\frac{61}{1296}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{61}{1296}\left|f'(b)\right|^q+\frac{29}{1296}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right\},$$

which is the same of the inequality in Theorem 10 in [11] for $s = 1$.

Corollary 2.3. *Under the assumptions of Theorem 2 with $\mu = \frac{1}{2}$ and $\lambda = 1$, we have the following trapezoid type inequality*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right| \leq (mb-a)\left(\frac{1}{8}\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\frac{1}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\left|f'(a)\right|^q\right.\right.$$

$$\left.+m\left(\frac{1}{8}-\frac{1}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{\alpha 2^{\alpha+1}+1}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\left|f'(a)\right|^q\right.$$

$$\left.\left.+m\left(\frac{1}{8}-\frac{\alpha 2^{\alpha+1}+1}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}.$$

Corollary 2.4. *Under the assumptions of Theorem 2 with $\mu = \frac{1}{2}$ and $\lambda = 0$, we have the following midpoint type inequality*

$$\left|f\left(\frac{a+mb}{2}\right)-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right|$$

$$\leq (mb-a)\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\times\left\{\left(\frac{1}{2^{\alpha+2}(\alpha+2)}\left|f'(a)\right|^q\right.\right.$$

$$\left.+m\left(\frac{1}{8}-\frac{1}{2^{\alpha+2}(\alpha+2)}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{2^{\alpha+2}-\alpha-3}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\left|f'(a)\right|^q\right.$$

$$\left.\left.+m\left(\frac{1}{8}-\frac{2^{\alpha+2}-\alpha-3}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}$$

Theorem 3. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a < b$ and $\lambda,\mu \in [0,1]$. If $\left|f'\right|^q$ is (α,m)-convex on $[a,b]$, for $(\alpha,m) \in (0,1]^2$, $mb > a$, $q > 1$, then the following inequality holds:*

$$\left|\lambda\left(\mu f(a)+(1-\mu)f(mb)\right)+(1-\lambda)f(\mu a+m(1-\mu)b)\right.$$

$$\left.-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right| \tag{10}$$

$$\leq (mb-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}.$$

$$\begin{cases}\left[\vartheta_1^{\frac{1}{p}}A^{\frac{1}{q}}+\vartheta_3^{\frac{1}{p}}B^{\frac{1}{q}}\right], & \lambda(1-\mu)\leq\mu\leq 1-\lambda\mu\\ \left[\vartheta_2^{\frac{1}{p}}A^{\frac{1}{q}}+\vartheta_3^{\frac{1}{p}}B^{\frac{1}{q}}\right], & \mu\leq\lambda(1-\mu)\leq 1-\lambda\mu\\ \left[\vartheta_2^{\frac{1}{p}}A^{\frac{1}{q}}+\vartheta_4^{\frac{1}{p}}B^{\frac{1}{q}}\right], & \lambda(1-\mu)\leq 1-\lambda\mu\leq\mu,\end{cases}$$

where

$$A=\mu\times\min\left\{\frac{\left|f'\left(\mu a+m\left(1-\mu\right)b\right)\right|^{q}+\alpha m\left|f'\left(b\right)\right|^{q}}{\alpha+1},\right.$$
$$\left.\frac{\left|f'\left(mb\right)\right|^{q}+\alpha m\left|f'\left(\frac{\mu a+m(1-\mu)b}{m}\right)\right|^{q}}{\alpha+1}\right\}$$

$$B=\left(1-\mu\right)\times\min\left\{\frac{\left|f'\left(\mu a+m\left(1-\mu\right)b\right)\right|^{q}+\alpha m\left|f'\left(\frac{a}{m}\right)\right|^{q}}{\alpha+1},\right.$$
$$\left.\frac{\left|f'\left(a\right)\right|^{q}+\alpha m\left|f'\left(\frac{\mu a+m(1-\mu)b}{m}\right)\right|^{q}}{\alpha+1}\right\}$$

$$\vartheta_1=\left[\lambda\left(1-\mu\right)\right]^{p+1}+\left[\mu-\lambda\left(1-\mu\right)\right]^{p+1},$$
$$\vartheta_2=\left[\lambda\left(1-\mu\right)\right]^{p+1}-\left[\lambda\left(1-\mu\right)-\mu\right]^{p+1},$$
$$\vartheta_3=\left[1-\lambda\mu-\mu\right]^{p+1}+\left[\lambda\mu\right]^{p+1},$$
$$\vartheta_4=\left[\lambda\mu\right]^{p+1}-\left[\mu-1+\lambda\mu\right]^{p+1},$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and by Hölder's integral inequality, we have

$$\left|\lambda\left(\mu f(a)+\left(1-\mu\right)f(mb)\right)+\left(1-\lambda\right)f(\mu a+m\left(1-\mu\right)b)\right.$$
$$\left.-\frac{1}{mb-a}\int_a^{mb}f(x)dx\right|$$
$$\leq\left(mb-a\right)\left[\int_0^{\mu}\left|-t+\lambda\left(1-\mu\right)\right|\left|f'\left(ta+m(1-t)b\right)\right|dt\right.$$
$$\left.+\int_{\mu}^{1}\left|-t+\left(1-\mu\lambda\right)\right|\left|f'\left(ta+m(1-t)b\right)\right|dt\right]$$
$$\leq\left(mb-a\right)\left\{\left(\int_0^{\mu}\left|-t+\lambda\left(1-\mu\right)\right|^{p}dt\right)^{\frac{1}{p}}\right.$$
$$\times\left(\int_0^{\mu}\left|f'\left(ta+m(1-t)b\right)\right|^{q}dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\mu}^{1}\left|-t+\left(1-\lambda\mu\right)\right|^{p}dt\right)^{\frac{1}{p}}$$
$$\left.\times\left(\int_{\mu}^{1}\left|f'\left(ta+m(1-t)b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right\}\tag{11}$$

Since $\left|f'\right|^{q}$ is (α,m)-convex on $[a,b]$, for $(\alpha,m)\in(0,1]^2$ and $\mu\in(0,1]$ by the inequality (2), we get

$$\int_0^{\mu}\left|f'\left(ta+m(1-t)b\right)\right|^{q}dt$$
$$=\mu\left[\frac{1}{\mu\left(mb-a\right)}\int_{\mu a+m(1-\mu)b}^{mb}\left|f'\left(x\right)\right|^{q}dx\right]$$
$$\leq\mu\times\min\left\{\frac{\left|f'\left(\mu a+m\left(1-\mu\right)b\right)\right|^{q}+\alpha m\left|f'\left(b\right)\right|^{q}}{\alpha+1},\right.$$
$$\left.\frac{\left|f'\left(mb\right)\right|^{q}+\alpha m\left|f'\left(\frac{\mu a+m(1-\mu)b}{m}\right)\right|^{q}}{\alpha+1}\right\}.\tag{12}$$

The inequality (12) also holds for $\mu=0$. Similarly, for $\mu\in[0,1)$ by the inequality (2), we have

$$\int_{\mu}^{1}\left|f'\left(ta+m(1-t)b\right)\right|^{q}dt$$
$$=\left(1-\mu\right)\left[\frac{1}{\left(1-\mu\right)\left(mb-a\right)}\int_a^{\mu a+m(1-\mu)b}\left|f'\left(x\right)\right|^{q}dx\right]$$
$$\leq\left(1-\mu\right)\times\min\left\{\frac{\left|f'\left(\mu a+m\left(1-\mu\right)b\right)\right|^{q}+\alpha m\left|f'\left(\frac{a}{m}\right)\right|^{q}}{\alpha+1},\right.$$
$$\left.\frac{\left|f'\left(a\right)\right|^{q}+\alpha m\left|f'\left(\frac{\mu a+m(1-\mu)b}{m}\right)\right|^{q}}{\alpha+1}\right\}.\tag{13}$$

The inequality (12) also holds for $\mu=1$. By simple computation

$$\int_0^{\mu}\left|-t+\lambda\left(1-\mu\right)\right|^{p}dt=\begin{cases}\frac{[\lambda(1-\mu)]^{p+1}+[\mu-\lambda(1-\mu)]^{p+1}}{p+1}, & \lambda\left(1-\mu\right)\leq\mu\\ \frac{[\lambda(1-\mu)]^{p+1}-[\lambda(1-\mu)-\mu]^{p+1}}{p+1}, & \lambda\left(1-\mu\right)\geq\mu\end{cases},\tag{14}$$

and

$$\int_{\mu}^{1}\left|-t+\left(1-\lambda\mu\right)\right|^{p}dt=\begin{cases}\frac{[1-\lambda\mu-\mu]^{p+1}+[\lambda\mu]^{p+1}}{p+1}, & \mu\leq1-\lambda\mu\\ \frac{[\lambda\mu]^{p+1}-[\mu-1+\lambda\mu]^{p+1}}{p+1}, & \mu\geq1-\lambda\mu\end{cases};\tag{15}$$

thus, using (12)-(15) in (11), we obtain the inequality (10). This completes the proof. □

Corollary 2.5. *Under the assumptions of Theorem 3 with $\mu = \frac{1}{2}$ and $\lambda = \frac{1}{3}$, we have the following Simpson type inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+mb}{2}\right)+f(mb)\right]-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right|$$

$$\leq\left(\frac{mb-a}{12}\right)\left(\frac{2^{p+1}+1}{3(p+1)}\right)^{\frac{1}{p}}\left(A_1^{\frac{1}{q}}+B_1^{\frac{1}{q}}\right),$$

where

$$A_1=\min\left\{\frac{\left|f'\left(\frac{a+mb}{2}\right)\right|^q+\alpha m\left|f'(b)\right|^q}{\alpha+1},\ \frac{\left|f'(mb)\right|^q+\alpha m\left|f'\left(\frac{a+mb}{2m}\right)\right|^q}{\alpha+1}\right\},$$

and

$$B_1=\min\left\{\frac{\left|f'\left(\frac{a+mb}{2}\right)\right|^q+\alpha m\left|f'\left(\frac{a}{m}\right)\right|^q}{\alpha+1},\ \frac{\left|f'(a)\right|^q+\alpha m\left|f'\left(\frac{a+mb}{2m}\right)\right|^q}{\alpha+1}\right\}.$$

Remark 3. In Corollary 2.5, if we take $\alpha = m = 1$, then we obtain the following Simpson type inequality

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^{b} f(x)dx\right|$$

$$\leq\left(\frac{b-a}{12}\right)\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\times\left\{\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\right\},$$

which is the same of the inequality in Corollary 3 in [11].

Corollary 2.6. *Under the assumptions of Theorem 3 with $\mu = \frac{1}{2}$ and $\lambda = 1$, we have the following trapezoid type inequality*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right|$$

$$\leq\left(\frac{mb-a}{4}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(A_1^{\frac{1}{q}}+B_1^{\frac{1}{q}}\right).$$

Corollary 2.7. *Under the assumptions of Theorem 3 with $\mu = \frac{1}{2}$ and $\lambda = 0$, we obtain the following midpoint type inequality*

$$\left|f\left(\frac{a+mb}{2}\right)-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right|$$

$$\leq\left(\frac{mb-a}{4}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(A_1^{\frac{1}{q}}+B_1^{\frac{1}{q}}\right).$$

Theorem 4. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$ and $\lambda, \mu \in [0,1]$. If $|f'|^q$ is (α, m)-convex on $[a,b]$, for $(\alpha,m) \in (0,1]^2$, $mb > a$, $q > 1$, then the following inequality holds:*

$$\left|\lambda\left(\mu f(a)+(1-\mu)f(mb)\right)+(1-\lambda)f(\mu a+m(1-\mu)b)-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right|$$

$$\leq(mb-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}.\left\{\begin{array}{l}\left[\vartheta_1^{\frac{1}{p}}A_2^{\frac{1}{q}}+\vartheta_3^{\frac{1}{p}}B_2^{\frac{1}{q}}\right],\ \lambda(1-\mu)\leq\mu\leq1-\lambda\mu\\ \left[\vartheta_2^{\frac{1}{p}}A_2^{\frac{1}{q}}+\vartheta_3^{\frac{1}{p}}B_2^{\frac{1}{q}}\right],\ \mu\leq\lambda(1-\mu)\leq1-\lambda\mu\\ \left[\vartheta_2^{\frac{1}{p}}A_2^{\frac{1}{q}}+\vartheta_4^{\frac{1}{p}}B_2^{\frac{1}{q}}\right],\ \lambda(1-\mu)\leq1-\lambda\mu\leq\mu\end{array}\right.,\tag{16}$$

where

$$A_2=\frac{\mu^{\alpha+1}\left|f'(a)\right|^q+m\left[\mu(\alpha+1)-\mu^{\alpha+1}\right]\left|f'(b)\right|^q}{\alpha+1}$$

and

$$B_2=\frac{\left(1-\mu^{\alpha+1}\right)\left|f'(a)\right|^q+m\left[\left(\mu^{\alpha+1}-1\right)+(1-\mu)(\alpha+1)\right]\left|f'(b)\right|^q}{\alpha+1}.$$

Proof. From Lemma 1 and by Hölder's integral inequality, we have inequality (11). Since $\left|f'\right|^q$ is (α, m)-convex on $[a,b]$, we know that for $t \in [0,\mu]$ and $t \in [\mu, 1]$

$$\left|f'\left(ta+m(1-t)b\right)\right|^q \le t^\alpha \left|f'(a)\right|^q + m(1-t^\alpha)\left|f'(b)\right|^q.$$

Hence,

$$\begin{aligned}
&\left|\lambda\left(\mu f(a)+(1-\mu)f(mb)\right)+(1-\lambda)f(\mu a+m(1-\mu)b)-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right| \\
&\le (mb-a)\left\{\left(\int_0^\mu \left|-t+\lambda(1-\mu)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^\mu \left[t^\alpha\left|f'(a)\right|^q+m(1-t^\alpha)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\int_\mu^1 \left|-t+(1-\alpha\lambda)\right|^p dt\right)^{\frac{1}{p}}\left(\int_\mu^1 \left[t^\alpha\left|f'(a)\right|^q+m(1-t^\alpha)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \qquad (17) \\
&\le (mb-a)\left\{\left(\int_0^\mu \left|-t+\lambda(1-\mu)\right|^p dt\right)^{\frac{1}{p}}\left(\frac{\mu^{\alpha+1}\left|f'(a)\right|^q+m\left[\mu(\alpha+1)-\mu^{\alpha+1}\right]\left|f'(b)\right|^q}{\alpha+1}\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\int_\mu^1 \left|-t+(1-\alpha\lambda)\right|^p dt\right)^{\frac{1}{p}}\left(\frac{\left(1-\mu^{\alpha+1}\right)\left|f'(a)\right|^q+m\left[\left(\mu^{\alpha+1}-1\right)+(1-\mu)(\alpha+1)\right]\left|f'(b)\right|^q}{\alpha+1}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Thus, using (14), (15) in (17), we obtain the inequality (16). This completes the proof. □

Corollary 2.8. *Let the assumptions of Theorem 4 hold. Then for $\mu=\frac{1}{2}$ and $\lambda=\frac{1}{3}$ from the inequality (16), we get the following Simpson type inequality*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+mb}{2}\right)+f(mb)\right]-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right| \\
&\le \left(\frac{mb-a}{12}\right)\left(\frac{2^{p+1}+1}{3(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2^\alpha(\alpha+1)}\right)^{\frac{1}{q}}\left(A_3^{\frac{1}{q}}+B_3^{\frac{1}{q}}\right).
\end{aligned}$$

where

$$A_3=\left|f'(a)\right|^q+m\left[2^\alpha(\alpha+1)-1\right]\left|f'(b)\right|^q$$

and

$$B_3=\left(2^{\alpha+1}-1\right)\left|f'(a)\right|^q+m\left[2^\alpha(\alpha+1)+1-2^{\alpha+1}\right]\left|f'(b)\right|^q.$$

Corollary 2.9. *Let the assumptions of Theorem 4 hold. Then for $\mu=\frac{1}{2}$ and $\lambda=1$ from the inequality (16), we get the following trapezoid type inequality:*

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right| \\
&\le \left(\frac{mb-a}{4}\right)\left(\frac{1}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2^\alpha(\alpha+1)}\right)^{\frac{1}{q}}\left(A_3^{\frac{1}{q}}+B_3^{\frac{1}{q}}\right).
\end{aligned}$$

Corollary 2.10. *In Corollary 2.9, if we take $\alpha=1$ we obtain the following trapezoid type inequality*

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right| \le (mb-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \\
&\times\left(\frac{1}{4}\right)^{1+\frac{1}{q}}\left[\left(\left|f'(a)\right|^q+3m\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(3\left|f'(a)\right|^q+m\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right] \qquad (18) \\
&\le (mb-a)\left(\frac{1}{4}\right)^{1+\frac{1}{q}}\left[\left(\left|f'(a)\right|^q+3m\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(3\left|f'(a)\right|^q+m\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where we have used the fact that $1/2<(1/(p+1))^{1/p}<1$. We note that the inequality (18) is the same of the inequality in Corollary 2.7 (i) [9].

Corollary 2.11. *Under the assumptions of Theorem 4 with $\mu=\frac{1}{2}$ and $\lambda=0$, we have the following midpoint type inequality:*

$$\begin{aligned}
&\left|f\left(\frac{a+mb}{2}\right)-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right| \\
&\le \left(\frac{mb-a}{4}\right)\left(\frac{1}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2^\alpha(\alpha+1)}\right)^{\frac{1}{q}}\left(A_3^{\frac{1}{q}}+B_3^{\frac{1}{q}}\right).
\end{aligned}$$

Competing interest
The author declares that he has no competing interest.

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